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Surface changes of temperature and matter due to coupled transport processes through porous media

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Abstract
A new calculation is elaborated for the description of surface changes of temperature and matter due to coupled transport through a porous medium. It is based on a finite-size supposition (Neumann's type boundary condition) and on the solutions of parabolic type partial differential equations combined with Lambert's W-function. The boundary layer phenomena are also incorporated into the description of the general transport. The procedure leads to a direct computer simulation, providing concrete results on the real physical picture of the given problem in good agreement with some experiments analysed.

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1. Introduction

As is widely experienced, the study of thermo-hydrodynamic effects plays a fundamental role equally from the points of view of both basic [1, 2] and applied [3] research. Accordingly, the aim of the present paper is to give an appropriate example for this topic. Thus, our investigation is dealing with a consequence of the coupled transport process through a porous medium, namely with the surface change of temperature and matter occurring due to its effect.

A general representation of the collective transport of characteristic extensive parameters is given by Onsagers' equations of the irreversible thermodynamics in the form

$$I_Q = L_{11}X_1 + L_{12}X_2 = L_{11}\nabla T + L_{12}\nabla\mu \tag{1}$$

$$I_m = L_{21}X_1 + L_{22}X_2 = L_{21}\nabla T + L_{22}\nabla\mu. \tag{2}$$

If the thermodynamic force X_1 means now the gradient of the temperature T and X_2 that of the chemical potential μ, then I_Q characterizes the total flux or current of the heat, while I_m

is that of the mass, in the creation of which the cross-effects also play a role. Among the elements of the conductivity L_{11} demonstrates pure heat transfer, L_{22} that of the mass, L_{12} the Duffour-effect and L_{21} the Soret-effect, respectively.

These relations provide a universal starting formalism [4] for the framework of the actual calculation of any physical processes, which are based on the transport of the heat and mass either by themselves or together.

Following the basic thermodynamic principles in the course of the description of the local temporal change of the mass density M and temperature T, we start from the system of partial differential equations (PDE) as

$$\frac{\partial M}{\partial t} = D\nabla^2 M + K\nabla^2 T \qquad \frac{\partial T}{\partial t} = \kappa\nabla^2 T + \gamma\nabla^2 M \tag{3}$$

where D is the diffusion coefficient, K is the thermodiffusion coefficient, κ is the heat conductivity and γ characterizes the magnitude of the change of the temperature originating from the local variation of the mass density at a given place.

In order to solve this PDE system, a simple finite-size, so-called zone-picture approximation is applied, supposing the existence of necessarily thin, but sufficiently thick layers (zones) inside which the macroscopic conductivity coefficients of the transport and coupled transport may be considered to have constant values. While applying equations (3) actually to some given problems, e.g. to the drying process, the relevant boundary conditions can be written in the rather general form of

$$\lambda_q(\nabla T)_b = F q_m(t) - q_q(t) \qquad \lambda_m(\nabla M)_b + G(\nabla T)_b = -q_m(t) \tag{4}$$

where the quantities F and G depend on the rate of the water vapour and liquid water, specific heat of possible phase transitions taking place inside the matter, respectively, while the fluxes of the heat and mass on the surface of the body are

$$q_q(t) = \alpha_q(T_0 - T_b) \qquad \text{and} \qquad q_m(t) = \alpha_m(M_b - M_p) \tag{5}$$

considering the convection, too. Here T_0 denotes the ambient temperature, T_b is the temperature on the boundary surface, $(M_b - M_p)$ is the difference of the values of the mass exchange potential between the boundary surface and the medium surrounding the drying body, where coupled heat and mass transfer occurs, α_q and α_m are the heat and the mass-exchange coefficients, respectively (particularly, in the case of general analysis of drying phenomena), $F = (1-\varepsilon)C_e$ and $G = \lambda_m\delta$, where ε is determined by the quantity of moving vapour in the moist body in relation to the total current of the vapour and liquid, C_e is the specific heat of evaporation, λ_m and λ_q are the coefficients of the mass- and heat-conductivity, respectively, and δ denotes the thermal gradient of the transfer of vapour [3]. In contrast to [5], where no particular boundary conditions were discussed, the general solution of system (3) can be derived analytically by simple application of the usual Fourier transformation method and presented as

$$\begin{aligned} M(x,t) &= \left\{\frac{1}{\sqrt{B}}\left[KC_2 + \frac{C_1}{2}(D-\kappa)\right] + \frac{C_1}{2}\right\}[\varphi_1(x,t) - \varphi_2(x,t)] \\ T(x,t) &= \left\{\frac{1}{\sqrt{B}}\left[\gamma C_1 - \frac{C_2}{2}(D-\kappa)\right] + \frac{C_2}{2}\right\}[\varphi_1(x,t) - \varphi_2(x,t)] \end{aligned} \tag{6}$$

where the values of the quantities C_1 and C_2 introduced due to the integration depend on the actual initial conditions and

$$\begin{aligned}
\varphi_1(x,t) &\equiv \frac{\exp\left(-\frac{x^2}{2t\sqrt{D+\kappa+\sqrt{B}}}\right)}{2t\sqrt{D+\kappa+\sqrt{B}}}\\
\varphi_2(x,t) &\equiv \frac{\exp\left(-\frac{x^2}{2t\sqrt{D+\kappa-\sqrt{B}}}\right)}{2t\sqrt{D+\kappa-\sqrt{B}}}\\
B &\equiv D^2 - 2D\kappa + \kappa^2 + 4K\gamma.
\end{aligned}\tag{7}$$

A specific property of these formulae is that they contain the time outside the square root of the denominators. These functionals differ from those separate PDEs, which characterize the usual mathematical formalism of the diffusion and/or heat conduction processes and have the advantage of leading to a new, complex analytical solution for the temperature level and moisture level, as shown in the following section.

2. Analysis of boundary conditions

2.1. General solution

In order to perform an adequate analysis of the problem for the above given boundary conditions we start from some effective methods widely applied in hydrodynamics [1]. For the sake of simplicity, we consider a one-dimensional problem and identify the boundary between two different continua with the (vertical) plane $x = 0$ considering a Neumann-type boundary condition for a parabolic-type transport equation for T:

$$-\lambda\left(\frac{\partial T}{\partial x}\right)_{x=0} = q(t) \qquad T = 0 \qquad t = -\infty \qquad x > 0 \tag{8}$$

where $q(t)$ denotes a known function of time and λ represents a general symbol of the heat-conductivity coefficient. It can then be shown that the solution of the conductivity problem is given in the following form

$$\lambda T(x,t) = \int_{-\infty}^{t} \sqrt{\frac{\chi}{\pi(t-\tau)}}\, q(\tau)\exp\left(-\frac{x^2}{4\chi(t-\tau)}\right)\mathrm{d}\tau \tag{9}$$

where χ denotes a quantity proportional to the heat conductivity coefficient and is usually called the heat propagation coefficient [3]. Then, if we consider conditions which give dependence of the temperature and mass transfer function on the boundary, i.e.

$$T(x,t) \to T(0,t) \equiv T_0(t) \qquad (T = 0, t \to -\infty, x > 0) \tag{10}$$

the solutions of the temperature and heat flux functions can be presented as follows:

$$T(x,t) = \frac{x}{2\sqrt{\pi\chi}} \int_{-\infty}^{t} \frac{T_0(\tau)}{(t-\tau)^{3/2}} \exp\left(-\frac{x^2}{4\chi(t-\tau)}\right)\mathrm{d}\tau$$

and

$$q_q(t) = \frac{\lambda}{\sqrt{\pi\chi}} \int_{-\infty}^{t} \frac{\mathrm{d}T_0(\tau)}{\mathrm{d}\tau} \frac{\mathrm{d}\tau}{\sqrt{t-\tau}}. \tag{11}$$

In the present case of the coupled transport problem it seems to be suitable for the investigation. Thus, the boundary condition discussed above can be generalized as

$$\begin{aligned}
\lambda_q(\nabla T)_b &= \alpha_q(T_0 - T_b) - F\alpha_m(M_b - M_p)\\
\lambda_m(\nabla M)_b &= -G\alpha_q(T_0 - T_b) + [GF - 1]\alpha_m(M_b - M_p)
\end{aligned}\tag{12}$$

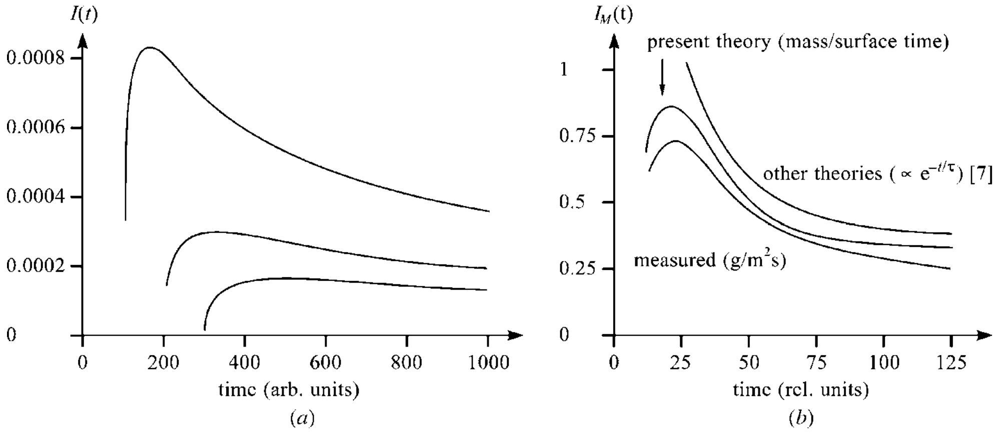

Figure 1. (*a*) Graphic presentation of the solution of the integral $I(t)$ (in relative units) and (*b*) an experimentally determined surface mass flux versus time.

and by identifying the boundary surface with the plane $x = 0$, we will obviously have

$$[\varphi_1(x=0,t) - \varphi_2(x=0,t)] = \frac{\sqrt{D+\kappa-\sqrt{B}} - \sqrt{D+\kappa+\sqrt{B}}}{4t\sqrt{D\kappa - K\gamma}}. \qquad (13)$$

Therefore, the equation

$$\alpha_q(T_0 - T_b) - F\alpha_m(M_b - M_p) = \left\{\frac{1}{\sqrt{B}}\left[\gamma C_1 - \frac{C_2}{2}(D-\kappa)\right] + \frac{C_2}{2}\right\} \times \frac{\sqrt{D+\kappa+\sqrt{B}} - \sqrt{D+\kappa-\sqrt{B}}}{4\sqrt{D\kappa - K\gamma}} \frac{\lambda_{\mathrm{eff}}^T}{\sqrt{\pi\chi_{\mathrm{eff}}^T}} \int_{-\infty}^{t} \frac{\mathrm{d}\tau}{\tau^2\sqrt{t-\tau}} \qquad (14)$$

can be obtained for the heat flux and the similar relation

$$G\alpha_q(T_c - T_b) + [GF - 1]\alpha_m(M_b - M_p) = \left\{\frac{1}{\sqrt{B}}\left[KC_2 + \frac{C_1}{2}(D-\kappa)\right] + \frac{C_1}{2}\right\} \times \frac{\sqrt{D+\kappa+\sqrt{B}} - \sqrt{D+\kappa-\sqrt{B}}}{4\sqrt{D\kappa - K\gamma}} \frac{\lambda_{\mathrm{eff}}^M}{\sqrt{\pi\chi_{\mathrm{eff}}^M}} \int_{-\infty}^{t} \frac{\mathrm{d}\tau}{\tau^2\sqrt{t-\tau}} \qquad (15)$$

for the mass flux. On the basis of former experience, some effective values of the heat conductivity and heat propagation coefficients $\lambda_{\mathrm{eff}}^{(T,M)}$ and $\chi_{\mathrm{eff}}^{(T,M)}$ must be used instead of simple coefficients relevant for separate heat (λ_q, χ_q) and mass (λ_m, χ_m) transfer, because these coefficients now correspond to a *coupled* transport process. Let us apply at present the function symbol $I(t)$ for the time-dependence expressed by the same integral in the above-given expressions. According to our knowledge, it represents a new solution formula for the theory of the coupled process, given by collective mass and heat transfer, which exists, e.g., in the course of a genuine drying procedure

$$I(t) = \int_{-\infty}^{t} \frac{\mathrm{d}\tau}{\tau^2\sqrt{t-\tau}} = \lim_{\tau\to(-\infty)} 2\frac{\sqrt{t-\tau}}{t\tau} - \frac{\frac{\sqrt{t-\tau}}{\tau} - \frac{\arctan h(\sqrt{t-\tau}/\sqrt{t})}{\sqrt{t}}}{t}. \qquad (16)$$

This function can be demonstrated graphically in figure 1(*a*), together with a graphical presentation of an experimental result [6] in figure 1(*b*). This experimental curve significantly differs from the usual theoretical curves (also presented in the same figure) having the character of exponential decay characterized by adequate relaxation time constants (see e.g. [7]). According to the evaluation of these measurements performed on artificial porous matter there is an inertial effect during drying illustrated by an increase in the moisture level at the beginning of the coupled-transport process. As is clearly seen, $I(t)$ leads to complete agreement with the experiments fulfilled, by taking different initial moments of positive values. Then, we may state with confidence that the process consists of two parts. First, the layer of adsorbed moisture (in the form of a moisture film) must be removed. (Until this subprocess ends, the moisture matter will accumulate in the regions near the boundary surface due to the diffusion of the moisture from the bulk of the porous matter in the direction of the boundary surface.) After this, the diffusion processes will dominate (resulting in the usual simple exponentially decaying part of the solution curve), since the adsorbed layer of moisture has already been removed.

Besides, at present we introduce some simplifying abbreviations for the coefficients as follows:

$$S_T = \left\{ \frac{1}{\sqrt{B}} \left[\gamma C_1 - \frac{C_2}{2}(D-\kappa) \right] + \frac{C_2}{2} \right\} \frac{\sqrt{D+\kappa+\sqrt{B}} - \sqrt{D+\kappa-\sqrt{B}}}{4\sqrt{D\kappa - K\gamma}} \frac{\lambda_{\text{eff}}^T}{\sqrt{\pi \chi_{\text{eff}}^T}} \tag{17}$$

$$S_M = \left\{ \frac{1}{\sqrt{B}} \left[K C_2 + \frac{C_1}{2}(D-\kappa) \right] + \frac{C_1}{2} \right\} \frac{\sqrt{D+\kappa+\sqrt{B}} - \sqrt{D+\kappa-\sqrt{B}}}{4\sqrt{D\kappa - K\gamma}} \frac{\lambda_{\text{eff}}^M}{\sqrt{\pi \chi_{\text{eff}}^M}}. \tag{18}$$

Then, with their help we arrive at a simple ordinary algebraic system of equations, which can be solved directly, and after some elementary algebraic operations these solutions have the form

$$\begin{aligned} T_b &= T_0 + \frac{[C_e(1-\varepsilon)(S_T\lambda_m\delta + S_M) - S_T]\alpha_m}{\alpha_m[\alpha_q - (1-\varepsilon)\lambda_m\delta(\alpha_q + a_q)]} I(t) \\ M_b &= M_p + \frac{\alpha_q(S_T\lambda_m\delta - S_M)}{\alpha_m[\alpha_q - (1-\varepsilon)\lambda_m\delta(\alpha_q + a_q)]} I(t). \end{aligned} \tag{19}$$

It means that in the case of drying processes these solutions depend on the liquid–vapour rate ε in a rather complicated, nonlinear way. Therefore, the frequently emphasized [2, 4] quasi-linear or even nonlinear character of the thermodynamical state-dependence of conductivity and coupling coefficients can be presented explicitly by use of tools of the formalism developed for practical engineering applications (e.g. in [3]).

2.2. Application of the boundary layer theory

In order to present the above-discussed modelling method in as general form as possible, in this section we demonstrate a possibility for its common application together with certain results of the boundary layer theory. We consider now the situation when the boundary surface is not normal to the inlet current density vector, but has a general position. Accordingly, the full incoming heat current density $\vec{q}_{\text{tot}}(t)$can be divided into two parts:

$$\vec{q}_{\text{tot}}(t) = \vec{q}(t) + \vec{q}_{\tan}(t) \tag{20}$$

i.e. one part $\vec{q}(t)$ can be treated in the same way as has been done in the previous section, and its tangential component $\vec{q}_{\tan}(t)$ within the framework of heat exchange phenomena taking place in the boundary layer. The temperature change of the surface due to the tangential part of the impinging heat flow current can be modelled by the common use of tools of the boundary layer theory and simple theory of the convective flow [8]. Let us apply these consideration results here directly. During this operation, the primed letters correspond to a coordinate system whose horizontal axis (x') lies parallel to the surface of the matter being dried, while the coordinate y' demonstrates another axis, perpendicular to this surface. Therefore, the actual equations are

$$\begin{aligned} &v_{x'}\frac{\partial v_{x'}}{\partial x'} + v_{y'}\frac{\partial v_{x'}}{\partial y'} = \nu\frac{\partial^2 v_{x'}}{\partial y'^2} + g_p\beta(T - T_0) \\ &v_{x'}\frac{\partial T}{\partial x'} + v_{y'}\frac{\partial T}{\partial y'} = \chi\frac{\partial^2 T}{\partial y'^2} \qquad \frac{\partial v_{x'}}{\partial x'} + \frac{\partial v_{y'}}{\partial y'} = 0 \end{aligned} \tag{21}$$

applied with boundary conditions, as

$$\begin{aligned} &\nu_{x'}(y' = 0) = \nu_{y'}(y' = 0) = 0 \qquad && \nu_{x'}(y' \to +\infty) = 0 \\ &T(y' = 0) = T_1 && T(y' \to +\infty) = T_0 \end{aligned} \tag{22}$$

where T_1 denotes the temperature of the plate, g_p is the actual acceleration due to the gravitational field, ν denotes the kinematic viscosity coefficient, $\chi = \lambda_a/\rho c_p$, where λ_a is the heat conductivity coefficient of air and β is the heat expansion coefficient of the air in the present case. The heat flux due to the boundary layer effect is

$$q_p = -\frac{1}{L}\int_0^L \lambda_a \left(\frac{\partial T}{\partial y'}\right)_{y'=0} \mathrm{d}x' = -\frac{4\lambda_a}{3}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\vartheta\right)(0, P)\, C_T\,(T_1 - T_0)\frac{1}{\sqrt[4]{L}} \tag{23}$$

where

$$C_T = \left[\frac{g_p\beta(T_1 - T_0)}{4\nu^2}\right]^{1/4}$$

L is the thickness of the body being dried, P is the Prandtl number and usually the function $\vartheta(\xi) = (T - T_0)/(T - T_1)$, where $\xi = C_T y'/\sqrt[4]{x'}$ is introduced in such types of calculations [1, 8], i.e. knowledge of its explicit form is of crucial importance. Another function to be introduced for solving this problem is $\varphi(\chi)$ defined via $v_x = 4\nu C_T^2\sqrt{x}\,\mathrm{d}\varphi(\xi)/\mathrm{d}\xi$. The system of *ordinary* differential equations (ODE) concerning this problem can be solved by use of a series technique[5]. After some allowed simplifications (presented in the appendix), an approximate solution of it can be written as

$$\vartheta'(0, P) = K_1\,\mathrm{e}^{-3PK_2} \qquad \text{where} \qquad K_1, K_2 = \text{const}. \tag{24}$$

At the same time, the boundary layer phenomenon can be described by an accurate calculation of the surface change of the temperature and moisture level. In this way, we have the following modification of the initial equation system, expressed by

$$\begin{aligned} &\lambda_q(\nabla T)_b = \left(\alpha_q + \frac{4\lambda_{\text{eff}}^T}{3}K_1\,\mathrm{e}^{-3PK_2}\right)(T_0 - T_b) - F\alpha_m(M_b - M_p) \\ &\lambda_m(\nabla M)_b = -G\left(\alpha_q + \frac{4\lambda_{\text{eff}}^T}{3}K_1\,\mathrm{e}^{-3PK_2}\right)(T_0 - T_b) + [GF - 1]\alpha_m(M_b - M_p) \end{aligned} \tag{25}$$

i.e. the coefficients appearing in the final solution of (12) will undergo a change, as

$$\alpha_q \to \left(\alpha_q + \frac{4\lambda_{\text{eff}}^T}{3}K_1\,\mathrm{e}^{-3PK_2}\frac{C_T}{\sqrt[4]{L}}\right). \tag{26}$$

Finally, let us consider the following modification of the previously explained idealized model, which has been related to bulk porous materials, being dried with ideally flat surfaces.

[5] MAPLE 8, a Symbolic Computation System, Waterloo Maple Inc. (2002).

Since solution (23) is valid in the case of geometrical conditions, when the thickness of the porous bulk matter is much larger than the magnitude of the thickness of the boundary layer and/or at sufficiently high values of the Grashof number G (i.e. when the condition $\sqrt[4]{G} \gg 1$ is satisfied corresponding to turbulent heat flow [1]), the above developed model calls for further refinements. Then, in such more refined descriptions of the turbulent boundary layer phenomena the roughness of the surfaces must also be taken into account. This refinement can be performed by using the tools of classical hydrodynamics and some specific complex functions, and the additional heat flux expression may be included subsequently in the general set of formulae discussed above. Starting from certain known relations developed within the framework of the boundary layer theory, we modify them by taking into account the order of magnitude of knobs placed on the surface of the matter being dried. In this case, the heat flux contribution [1] in the turbulent boundary layer is

$$\vec{q}_{\mathrm{blturb}} = \rho c_p \chi_{\mathrm{turb}} (\nabla T)_{\mathrm{bl}} \tag{27}$$

where it is assumed that the flowing fluid is incompressible. It must be emphasized that the temperature gradient in the case of turbulent boundary layer heat conduction$(\nabla T)_{\mathrm{bl}}$ is related directly to the boundary layer itself and therefore differs from $(\nabla T)_b$. In order to estimate the heat flow contribution due to boundary layer effects, we take into account here the relationship $\chi_{\mathrm{turb}} \propto \frac{1}{\nu^3}(y v^*)^4$ valid for high values of the Prandtl number (in this expression v^* denotes the characteristic velocity of the turbulent flow, having the explicit form $v^* = U\sqrt{\frac{c}{2}}$ [1]). Therefore, the heat propagation coefficient for such flows is given as

$$\chi_{\mathrm{turb}} \propto \frac{y^4 U^4}{\nu^3} c^2 \tag{28}$$

where y denotes the distance from the surface of the solid body, surrounded by turbulent flow (at least in the boundary layer) of an incompressible fluid, U is the constant velocity of the basic fluid flow and c is the flow resistance factor. The *average* air flow resistance coefficient C can be calculated rather accurately, if we know the order of magnitude of knobs d on the surface of the body being dried. Namely, within the framework of the boundary layer theory, this factor is defined as

$$C = \frac{1}{l} \int_0^l c'(x')\,\mathrm{d}x' \tag{29}$$

where l means the length of the sample, and $c(x)$ obeys the following relation (whose validity can be accepted with logarithmic accuracy):

$$\frac{0.59}{\sqrt{c}} = \ln \frac{x'\sqrt{c}}{d} \tag{30}$$

where the length x' is measured along the surface of the bulk porous matter. The solution of this equation with respect to $c(x')$ can be explained by Lambert's W-function (precisely with respect to its analytical branch [9]), which we found by use of the MAPLE software package (see footnote 5).

Therefore, the resistance coefficient is given by the integral of

$$C = \int_0^l \mathrm{Lamb}\, W(d, x')\,\mathrm{d}x' = \frac{l[1 - \mathrm{Lamb}\, W(d, l) + \mathrm{Lamb}\, W^2(d, l)]}{\mathrm{Lamb}\, W(d, l)} \tag{31}$$

i.e. the solution is explained by use of the Lambert W-function. The graphical presentation of the resistance coefficient in relative units is given in figures 2(a) and (b).

These results are in good qualitative agreement with certain earlier ones obtained by variational principles of the extended irreversible thermodynamics [10], moreover with some

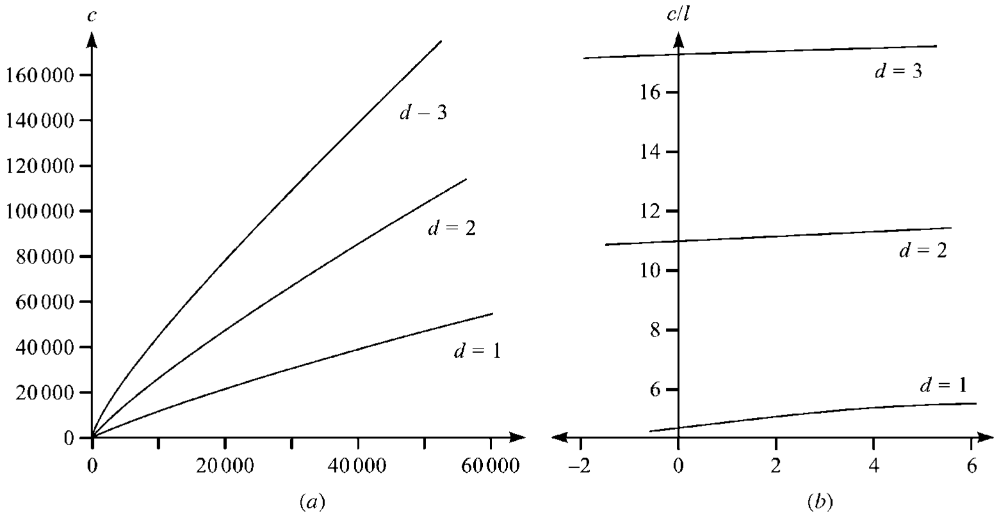

Figure 2. Shape of the air resistance factor according to the calculations based on the use of the Lambert W-function: (*a*) solution curves of the air resistance obtained by direct application of the formula (31) for three different possible magnitudes of knobs, (*b*) air resistances calculated for unit length in the case of the same magnitudes of knobs.

drying experiments [6]. This fact demonstrates the usefulness of the application of the Lambert W-function to solve the task of the change of surface temperature and matter during a coupled transport process.

3. Conclusions

1. As is seen, the calculation given in this paper can describe the basic complex phenomena of the coupled transport effects, which lead to the creation of surface inhomogeneity of temperature and matter.
2. This method can contain expressively the thermodynamic cross-effects, which play the main role in the development of surface changes of temperature and matter, but does not reflect directly the dissipative character of the phenomenon.
3. A recent work, dealing with a refined model of the properties of the surface moisture level [11] hints at the possibility of a claim to investigate the problem beyond the classical irreversible thermodynamic theory. We believe that our present research, which points to a new method, based on the application of the Lambert W-function, can supplement efficiently this idea and that the common continuation of both will provide some more precise information on the questions analysed above.

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Appendix

In this appendix the calculation leading to the final solution (24) will be presented briefly. As is well known from the boundary layer theory of the viscous flow, the problem discussed by Pohlhausen [8] leads to the following nonlinear ODE system (in all of the forthcoming equations the primes denote derivatives with respect to the variable ζ):

$$\varphi''' + 3\varphi\varphi'' - 2\varphi'^2 + \vartheta = 0 \qquad \vartheta'' + 3P\vartheta' = 0 \tag{32}$$

with boundary conditions:

$$\varphi(0) = 0 \qquad \varphi'(0) = 0 \qquad \vartheta(0) = 0 \qquad \varphi'(\infty) = 0 \qquad \vartheta(\infty) = 0. \tag{33}$$

We solved this coupled nonlinear system by a series method in a relatively low-order approximation using the MAPLE software package (see footnote 5). The solutions are:

$$\begin{aligned}
\varphi(\xi) = {} & \varphi(0) + \left.\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right|_{\xi=0} \xi + \frac{1}{2}\left.\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2}\right|_{\xi=0} \xi^2 - \left(\frac{1}{2}\varphi(0)\left.\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2}\right|_{\xi=0} + \frac{1}{6}\vartheta(0)\right)\xi^3 \\
& + \left(\frac{3}{8}(\varphi(0))^2 \left.\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2}\right|_{\xi=0} + \frac{1}{8}\varphi(0)\vartheta(0) - \frac{1}{8}\left.\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right|_{\xi=0}\left.\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2}\right|_{\xi=0} - \frac{1}{24}\left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi=0}\right)\xi^4 \\
& + \left(-\frac{9}{40}(\varphi(0))^3 \left.\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2}\right|_{\xi=0} - \frac{3}{40}(\varphi(0))^2\vartheta(0) + \frac{9}{40}\varphi(0)\left.\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right|_{\xi=0}\left.\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2}\right|_{\xi=0}\right. \\
& \left. + \frac{1}{40}\varphi(0)\left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi=0} + \frac{\left.\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right|_{\xi=0}\vartheta(0)}{20}\right)\xi^5 + \left(-\frac{1}{40}\left(\left.\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2}\right|_{\xi=0}\right)^2\right. \\
& \left. + \frac{P}{40}\varphi(0)\left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi=0}\right)\xi^5 + \mathrm{O}(\xi^6)
\end{aligned}$$

for the function $\varphi(\xi)$ and

$$\begin{aligned}
\vartheta(\xi) = {} & \vartheta(0) + \left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi=0} \xi - \frac{3P}{2}\varphi(0)\left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi=0}\xi^2 + \left(\frac{3}{2}P^2(\varphi(0))^2\left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi=0}\right. \\
& \left. - \frac{P}{2}\left.\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right|_{\xi=0}\left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi=0}\right)\xi^3 + \left(\frac{9}{8}P^2\varphi(0)\left.\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right|_{\xi=0}\left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi=0}\right. \\
& \left. - \frac{9}{8}P^3(\varphi(0))^3\left.\frac{\mathrm{d}\vartheta}{d\xi}\right|_{\xi=0} - \frac{1}{8}P\left.\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2}\right|_{\xi=0}\left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi=0}\right)\xi^4 \\
& + \left(\frac{3}{10}P^2\varphi(0)\left.\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2}\right|_{\xi=0}\left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi=0} + \frac{27}{10}P^4(\varphi(0))^4\left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi=0}\right. \\
& \left. - \frac{27}{20}P^3(\varphi(0))^2\left.\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right|_{\xi=0}\left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi=0}\right)\xi^5 + \left(\frac{9}{40}P^2\left(\left.\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right|_{\xi=0}\right)^2\left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi=0}\right. \\
& \left. + \frac{3}{40}P\varphi(0)\left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi=0}\left.\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2}\right|_{\xi=0} + \frac{1}{40}P\vartheta(0)\left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi=0}\right)\xi^5 + \mathrm{O}(\xi^6)
\end{aligned}$$

for the function $\vartheta(\xi)$. These solutions can be simplified directly and relatively easily by use of the relevant relationships from the boundary conditions (33), namely, by $\varphi(0) = 0$, $\varphi'(0) = 0$, $\vartheta(0) = 0$. After allowed simplifications, many terms in both solutions become annulated and $\vartheta'(0, P)$, playing a crucial role in expression (23), can be calculated by trivial integration methods leading to (24).

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